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EL_P ლოგიკის ამოხსნადობა

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Decidability of the Logic $E Ł_P$

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ABSTRACT

The new modal epistemic Łukasiewicz logic $EŁ_P$ is introduced, obtained from the infinitely valued Łukasiewicz logic *Ł* by adding one axiom of the logic $Ł_P$ of perfect *MV*-algebras, the language of which is enriched by "quasi knowledge operator" with corresponding axioms.

It is proved that the set of theorems of the logic $EŁ_P$ is recursively enumerable.

აბსტრაქტი

შემოღებულია ახალი მოდალური ეპისტემიკური ლუკასევიჩის ლოგიკა $EŁ_P$, რომელიც მიღებულია $Ł$ ლუკასევიჩის უსასრულო ნიშნა ლოგიკისგან სრულყოფილი *MV*-ალგებრის $Ł_P$ ლოგიკისგან ერთი აქსიომის დამატებით, რომლის ენა გამდიდრებულია კვაზი-ცოდნის ოპერატორით შესაბამისი აქსიომებით.

დამტკიცებულია, რომ $EŁ_P$ ლოგიკის თეორემათა სიმრავლე რეკურსიულად გადათვლადია.

Monadic MV-algebras

The finitely valued propositional calculi, which have been described by Łukasiewicz and Tarski in 1930, are extended to the corresponding predicate calculi.

The predicate Łukasiewicz (infinitely valued) logic *QL* is defined in standard way.

Monadic MV-algebras

Monadic *MV*-algebras were introduced and studied by Rutledge in

[**J.D. Rutledge, *A preliminary investigation of the infinitely many-valued predicate calculus*, Ph.D. Thesis, Cornell University, 1959**.]

as an algebraic model for the predicate calculus *QL* of Łukasiewicz infinite valued logic, in which only a single individual variable occurs.

Monadic Logic

Let L denote a first-order language based on $\cdot$, +, $\to$, $\neg$, $\exists$ and let L_m denote a propositional language based on $\cdot$, +, $\to$, $\neg$, $\exists$. Let $\mathrm{Form}(L)$ and $\mathrm{Form}(L_m)$ be the set of all formulas of L and L_m, respectively.

We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $p{*}(x)$ in L and denote by induction a translation

$$\Psi : \mathrm{Form}(L) \to \mathrm{Form}(L_m)$$ by putting:

- $\Psi(p) = p{*}(x)$ if p is propositional variable,
- $\Psi(\alpha \square \beta) = \Psi(\alpha) \square \Psi(\beta)$, where $\square \in \{\cdot, +, \to\}$,
- $\Psi(\exists\alpha) = \exists x\ \Psi(\alpha)$.

MV-algebras

An ***MV-algebra*** is an algebra

$$\mathrm{A} = (A, \oplus, \otimes, {}^*, 0, 1)$$

where $(A, \oplus, 0)$ is an *abelian monoid*, and for all $x, y \in A$ the following identities hold:

$x \oplus 1 = 1, \; x^{**} = x,$

$(x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x,$

$x \otimes y = (x^* \oplus y^*)^*.$

MV -algebras

It is well known that the *MV*-algebra S = ([0, 1], $\oplus$, $\otimes$, *, 0, 1), where $x \oplus y$ = min(1, x+y), $x \otimes y$ = max(0, x+y -1), x* = 1-x, generates the variety **MV** of all *MV*-algebras.

Let Q denote the set of rational numbers, for $(0 \neq)$ $n \in \omega$ we set S_n =(S_n, $\oplus$, $\otimes$, *, 0, 1), where S_n = {0, 1/n-1, … , n-2/n-1, 1} is also *MV*-algebra.

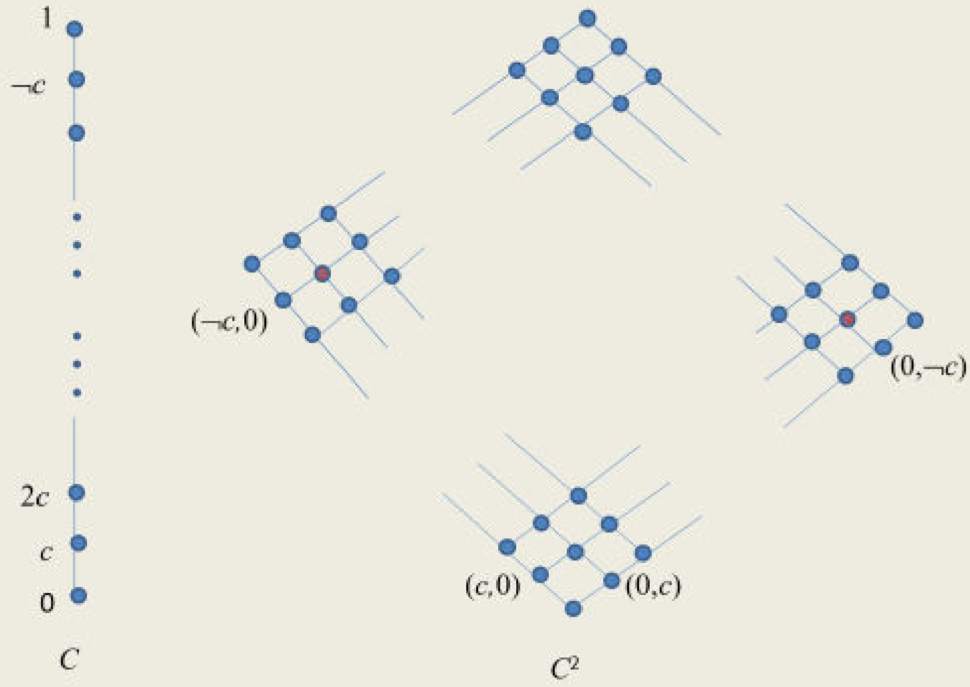
Perfect MV-algebras

From the variety of *MV*-algebras **MV** select the subvariety **MV(C)** which is defined by the following identity:

$$\text{(Perf)} \quad 2(x^2) = (2x)^2,$$

that is **MV(C)** = **MV**+ (Perf).

Perfect MV-algebras



Logic $Ł_P$

$Ł_P$ is the logic corresponding to the variety generated by perfect *MV*-algebras which coincides with the set of all Łukasiewicz formulas that are valid in all perfect *MV*-chains, or equivalently that are valid in the *MV*-algebra *C*. Actually, $Ł_P$ is the logic obtained by adding to the axioms of Łukasiewicz sentential calculus the following axiom:

$$Ł_P:\ (\alpha \underline{\vee}\ \alpha)\&(\alpha \underline{\vee}\ \alpha) \leftrightarrow (\alpha \,\&\, \alpha)\underline{\vee}(\alpha \,\&\, \alpha)$$

Logic $Ł_P$

Theorem. *A formula of the logic $Ł_P$ is a theorem iff it is 1-tautology in the algebra C.*

Monadic MV-algebras

An algebra A $=(A,\oplus, \otimes, {}^*, \exists, 0, 1)$ (also denoted as $(A,\exists)$) is said to be ***a monadic MV-algebra*** (for short *MMV*-algebra) [**A.Di Nola, R.Grigolia**] if $(A, \oplus, \otimes, {}^*, 0, 1)$ is an *MV*-algebra and in addition $\exists$ satisfies the following identities:

- E1. $x \leq \exists x$,
- E2. $\exists(x \vee y) = \exists x \vee \exists y$,
- E3. $\exists(\exists x)^* = (\exists x)^*$,
- E4. $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$,
- E5. $\exists(x \otimes x) = \exists x \otimes \exists x$,
- E6. $\exists(x \oplus x) = \exists x \oplus \exists x$.

Łukasiewicz logic

The original system of axioms for propositional infinite-valued Łukasiewicz logic used implication and negation as the primitive connectives as for classical logic:

L1. $(\alpha \to (\beta \to \alpha))$
L2. $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
L3. $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$
L4. $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$.

There is only one inference rule - Modus Ponens: from α and $(\alpha \to \beta)$ infer β.

Łukasiewicz logic

- **Theorem 2.** [Chang]. (Completeness theorem). *A Lukasiewicz formula* α *is a theorem iff* α *is a tautology.*

Logic $Ł_P$

$Ł_P$ is the logic obtained by adding to the axioms of Łukasiewicz sentential calculus the following axiom:

$$Ł_P\text{:}\ (\alpha \veebar \alpha)\&(\alpha \veebar \alpha) \leftrightarrow (\alpha \& \alpha)\veebar(\alpha \& \alpha)$$

Monadic Łukasiewicz Logic *MŁ*

Monadic Łukasiewicz propositional calculus *MŁ* as a logic which contains Lukasiewicz propositional calculus *Ł*, the formulas as the axiom schemas:

M1. $\Box^q\varphi \to \varphi$,

M2. $\Box^q (\varphi \wedge \psi) \leftrightarrow \Box^q\varphi \wedge \Box^q\psi$,

M3. $\Box^q(\neg \Box^q\varphi) \leftrightarrow \neg \Box^q\varphi$,

M4. $\Box^q (\Box^q\varphi \,\&\, \Box^q\psi) \leftrightarrow \Box^q\varphi \,\&\, \Box^q\psi$,

M5. $\Box^q(\varphi \,\&\, \varphi) \leftrightarrow \Box^q\varphi \,\&\, \Box^q \varphi$,

M6. $\Box^q(\varphi \underline{\vee} \varphi) \leftrightarrow \Box^q\varphi \underline{\vee} \Box^q \varphi$,

inference rules: $\varphi, \varphi \to \psi / \psi$, $\varphi / \Box^q\varphi$.

Monadic Łukasiewicz Logic *MŁ*

- **Theorem 3.** [Rutleddge; Di Nola, Grigolia]. *A modal formula* φ *is a theorem of MŁ if it is a tautology*

Logic $E Ł_P$

Modal Epistemic Lukasiewicz logic $E Ł_P$ is a logic which contains monadic Łukasiewicz propositional calculus $MŁ$ and the formula as the axiom scheme:

$$Ł_P: \ (\alpha \veebar \alpha) \& (\alpha \veebar \alpha) \leftrightarrow (\alpha \& \alpha) \veebar (\alpha \& \alpha)$$

Logic $EŁ_P$

Theorem 4. *A formula φ of $EŁ_P$ is a theorem if it is a tautology.*

Decidability of $EŁ_P$

In the sequel *Form(L)* denotes the set of all formulas of the logic *L* and *Th(L)* the set of all theorems of the logic L.

A set *X* is called *recursive* (or *decidable*) if there is an algorithm which, given an object *x* from the class under consideration, recognizes whether

$x \in X$ or not. *X* is said to be *recursively enumerable* if one of the following equivalent conditions is satisfied:

1) *X* is the domain of a partial recursive function;
2) *X* is either the range of a total recursive function or empty.

Decidability of $EŁ_P$

Proposition 1. *Suppose* Y *is a recursive set and* $X \subset Y$. *Then* X *is recursive iff both* X *and* $Y - X$ *are recursively enumerable.*

Decidability of $EŁ_P$

Proposition 1. *Suppose* Y *is a recursive set and* $X \subset Y$. *Then* X *is recursive iff both* X *and* $Y - X$ *are recursively enumerable.*

Proposition 2. [Chagrov and Zakharyaschev].

(1) *Form(L) is recursively enumerable (without repetitions). Moreover, these sets are recursive.*

(2) *The set Th(L) of theorems of a logic L with a recursively enumerable set of axioms is also recursively enumerable.*

Decidability of $EŁ_P$

- **Proposition 3.** (Craig's theorem) *For every logic L the following conditions are equivalent:*
- (i) *L has a recursively enumerable set of axioms;*
- (ii) *L has a recursive set of axioms;*
- (iii) *Th(L) is recursively enumerable.*

Decidability of $EŁ_P$

Proposition 4. [Chagrov and Zakharyaschev]. *If theorems of a logic L is characterized by a recursively enumerable class C of recursive algebras then the set of formulas that are not theorems in L is also recursively enumerable.*

Decidability of $EŁ_P$

Proposition 4. [Chagrov and Zakharyaschev]. *If theorems of a logic L is characterized by a recursively enumerable class C of recursive algebras then the set of formulas that are not theorems in L is also recursively enumerable.*

Proposition 5. [Chagrov and Zakharyaschev]. *A logic is decidable if it is recursively axiomatizable and charcharacterized by a recursively enumerable class of recursive algebras.*

Decidability of $EŁ_P$

Theorem 5. *The logic $EŁ_P$ is recursively axiomatizable and charcharacterized by a recursively enumerable class of recursive algebras*

Decidability of $EŁ_P$

Theorem 5. *The logic* $EŁ_P$ *is decidable.*

THANK YOU